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Optimal Control of Harvesting of a Distributed Renewable Resource on the Earth's Surface

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Abstract—This paper is devoted to the optimal control of mixed (stationary and periodic impulse) harvesting of a renewable resource distributed on the Earth's surface. Examples of such a resource are biological populations, including viruses, chemical contaminants, dust particles, and the like. It is proved that on an infinite planning horizon, there exists an admissible control ensuring the maximum of time-averaged harvesting.

Keywords: the Kolmogorov–Petrovskii–Piskunov–Fisher equations, second-order parabolic equations, semilinear equations on a sphere, weak solutions, stabilization, optimal control

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1. INTRODUCTION

Two-dimensional (2D) manifolds homeomorphic to a sphere are commonly used as a mathematical model of the Earth's surface. The dynamics of a renewable resource distributed on the Earth's surface can be modeled by a second-order semilinear evolutionary equation on a 2D sphere. In local coordinates, it has the form

$$\frac{\partial q}{\partial t} - \sum_{l,m=1}^{2} \frac{\partial}{\partial x^l}\left(a^{l,m}(x)\frac{\partial q}{\partial x^m}\right) = A(x)q - B(x)q^2, \quad a^{l,m}(x) = a^{m,l}(x), \tag{1}$$

where the matrix a characterizes the resource diffusion, and the coefficients A and B are the resource renewal and saturation rates of the environment. Essentially, equation (1) combines two classical models: the Verhulst logistic model [1] and the Fourier heat propagation model [2].

Equations of the form (1) arise when modeling various reaction–diffusion processes in a distributed environment. One example is the famous model proposed by A.N. Kolmogorov, G.I. Petrovskii, and N.S. Piskunov [3] and R.A. Fischer [4]. Information about other models, the history and bibliography of the works on this topic can be found in [5]. Also, the interested reader is referred to the monograph [6], covering several applied aspects.

Second-order semilinear evolutionary equations in Euclidean space domains have been studied quite thoroughly; for example, see [7–9]. On closed manifolds, particularly spheres, they have been investigated to a lesser extent. It is appropriate to mention the papers devoted to the equations with periodically fragmented coefficients [5, 10] (in fact, equations on a torus). This case, important from an applied point of view, occurs when modeling periodic media. Of course, equations of the form (1) on a 2D sphere are also of significant interest: this is a standard model of the Earth's surface used in applications.

Note that many applied problems lead to equations of the type (1) with discontinuous coefficients. In particular, this is characteristic of optimal control problems. Therefore, it is desirable to choose a class of admissible solutions to construct a satisfactory theory of the corresponding equations with minimal regularity requirements for their coefficients. In this paper, such a class consists of weak solutions. In the class of weak solutions, it is possible to study equations of the form (1) on a 2D sphere with fairly light regularity requirements for their coefficients.

2. FUNCTION SPACES AND EVOLUTIONARY EQUATIONS

2.1. Function Spaces

Let $\mathbb{S}^2$ be a 2D sphere of unit radius, $\{(y^1, y^2, y^3) \in \mathbb{R}^3 \mid (y^1)^2 + (y^2)^2 + (y^3)^2 = 1\}$, standardly embedded in the 3D Euclidean space $\mathbb{R}^3$. The stereographic projection

$$h : \mathbb{S}^2\backslash(0,0,1) \ni (y^1, y^2, y^3) \mapsto \frac{(y^1, y^2)}{1 - y^3} \in \mathbb{R}^2$$

relative to the pole $(0, 0, 1)$ specifies a local coordinate system defined on $\mathbb{S}^2$ everywhere except the pole [11] (lecture 6). An embedding in the Euclidean space $\mathbb{R}^3$ induces on $\mathbb{S}^2$ a Riemannian metric g, whose inverse image relative to h^{-1} has the form

$$(h^{-1})^* g = 4\frac{(dx^1)^2 + (dx^2)^2}{((x^1)^2 + (x^2)^2 + 1)^2}.$$

Here h^{-1} is a mapping inverse to the stereographic projection, i.e.,

$$h^{-1} : \mathbb{R}^2 \ni (x^1, x^2) \mapsto \frac{1}{(x^1)^2 + (x^2)^2 + 1}(2x^1, 2x^2, (x^1)^2 + (x^2)^2 - 1) \in \mathbb{S}^2. \tag{2}$$

The metric g defined on the tangent bundle $T\mathbb{S}^2$ admits a natural extension to tensor bundles $(T\mathbb{S}^2)^{\bigotimes^m} \bigotimes (T^*\mathbb{S}^2)^{\bigotimes^l}$, $m, l = 0, 1, 2, \ldots$, which will be denoted by the same symbol g. On $(T\mathbb{S}^2)^{\bigotimes^0} \bigotimes (T^*\mathbb{S}^2)^{\bigotimes^0} = \mathbb{S}^2 \times \mathbb{R}$, the metric is $g(r_1, r_2) = r_1 r_2$ for $r_1, r_2 \in \mathbb{R}$. Also, g induces on $\mathbb{S}^2$ a complete metric space structure and a measure $\mu = \mu_g$, whose image relative to the stereographic projection has the form

$$d(\mu \circ h) = \frac{4dx^1 dx^2}{((x^1)^2 + (x^2)^2 + 1)^2}. \tag{3}$$

These structures are used to build the Lebesgue spaces of functions and tensor fields, $L^p(\mathbb{S}^2)$ and $L^p((T\mathbb{S}^2)^{\bigotimes^m} \bigotimes (T^*\mathbb{S}^2)^{\bigotimes^l})$, where $p \geqslant 1$ and $m, l = 0, 1, 2, \ldots$, as well as the Sobolev spaces $W^{1,p}(\mathbb{S}^2)$ and $W^{1,p}((T\mathbb{S}^2)^{\bigotimes^m} \bigotimes (T^*\mathbb{S}^2)^{\bigotimes^l})$ [12, Ch. 2] and the Hölder spaces $C^\alpha(\mathbb{S}^2)$ and $C^\alpha((T\mathbb{S}^2)^{\bigotimes^m} \bigotimes (T^*\mathbb{S}^2)^{\bigotimes^l})$, $0 < \alpha \leqslant 1$ [13, Sec. 10.2.4; 14; 15; 16, §1]. For this purpose, the stereographic coordinates (2) can be applied. For example, the function spaces $L^p(\mathbb{S}^2)$ on a sphere and $L^p(\mathbb{R}^2, \mu \circ h)$ on the plane with the measure (3) are isometric for $p \geqslant 1$.

Consider real-valued measurable functions u and v defined on $\mathbb{S}^2$ and let

$$ess \sup_{x \in \mathbb{S}^2} u(x) = \inf_{\substack{S \subseteq \mathbb{S}^2,\\ \mu(S)=0}} \sup_{x \in \mathbb{S}^2 \backslash S} u(x),$$

$$ess \inf_{x \in \mathbb{S}^2} u(x) = \sup_{\substack{S \subseteq \mathbb{S}^2,\\ \mu(S)=0}} \inf_{x \in \mathbb{S}^2 \backslash S} u(x), \langle u, v \rangle = \int_{\mathbb{S}^2} uv d\mu.$$

If $\mathfrak{B}$ is a Banach space with a norm $\|\cdot\|_{\mathfrak{B}}$, then for fixed $T_0 \in (0,+\infty)$ and $T_1 \in (0,+\infty]$, $T_0 < T_1$, the spaces $L^p([T_0,T_1);\mathfrak{B})$ with the norms

$$\|q\|_{L^p([T_0,T_1);\mathfrak{B})} = \left(\int\limits_{T_0}^{T_1} \|q(t)\|_{\mathfrak{B}}^p dt\right)^{\frac{1}{p}}, \quad p \geqslant 1,$$

$$\|q\|_{L^\infty([T_0,T_1);\mathfrak{B})} = ess \sup_{t\in[T_0,T_1)} \|q(t)\|_{\mathfrak{B}}$$

are also Banach spaces; see [17, Ch. III, §1] and [18, Ch. II, §2]. The intersection

$$W([T_0,T_1);X) = L^2((T_0,T_1);W^{p,1}(X)) \cap L^\infty([T_0,T_1);L^2(X))$$

is also a Banach space with the norm

$$\|q\|^2_{W([T_0,T_1);X)} = ess \sup_{t\in[T_0,T_1)} \langle q(t),q(t)\rangle + \int\limits_{T_0}^{T_1} \langle g(dq(t),dq(t)),1\rangle dt.$$

For brevity, we will use the abbreviation *a.e.* whenever some properties are valid almost everywhere in the measure μ on $\mathbb{S}^2$, see (3).

2.2. Evolutionary Equations

Along with g, let another metric a be defined on the sphere $\mathbb{S}^2$. Assume that this metric is measurable and there exist $a_0, a_1 \in (0,+\infty)$ such that

$$a_0 g(\eta,\eta) \leqslant a(\eta,\eta) \leqslant a_1 g(\eta,\eta), \quad \eta \in T^*\mathbb{S}^2, \text{ a.e.} \tag{4}$$

In the stereographic coordinates x^1 and x^2 (2), the estimate (4) has the form

$$\frac{4a_0(\eta_1^2+\eta_2^2)}{((x^1)^2+(x^2)^2+1)^2} \leqslant a^{1,1}(t)\eta_1^2 + 2a^{1,2}(t)\eta_1\eta_2 + a^{2,2}(t)\eta_2^2 \leqslant \frac{4a_1(\eta_1^2+\eta_2^2)}{((x^1)^2+(x^2)^2+1)^2}.$$

Consider the differential operator $d^*_{a,g} : C^\infty(T^*\mathbb{S}^2) \ni w \mapsto d^*_{a,g} w \in C^\infty(\mathbb{S}^2)$ adjoint to the exterior differentiation operator d with respect to the metrics g and a, i.e.,

$$\langle a(du,\omega),1\rangle_g = \langle u, d^*_{a,g}\omega\rangle_g, \quad u \in C^\infty(\mathbb{S}^2), \ \ \omega \in C^\infty(T^*\mathbb{S}^2);$$

for details, see [19, Ch. VIII, §1]. In the system of the stereographic coordinates x^1 and x^2 (2),

$$d^*_{a(t),g}\omega = -((x^1)^2+(x^2)^2+1)^2 \sum_{l,m=1}^{2} \frac{\partial}{\partial x^l} \frac{a(dx^l,dx^m)}{((x^1)^2+(x^2)^2+1)^2}\, \omega\left(\frac{\partial}{\partial x^m}\right).$$

Given a function $u \in C^\infty(\mathbb{S}^2)$, we define the *geometric Laplacian* (*the Laplace–de Rahm operator*), i.e., the linear second-order differential operator [20, Ch. IV, §5]

$$\triangle = \triangle_{a,g} = d^*_{a,g} \circ d. \tag{5}$$

Due to the estimate (4), the operator (5) is *uniformly elliptic* on $\mathbb{S}^2$.

Hence, the second-order evolutionary equation

$$\frac{\partial q}{\partial t} + \triangle q = (A(x) - u(x))q - B(x)q^2 \tag{6}$$

is *parabolic* on $\mathbb{S}^2$. In the stereographic coordinates x^1 and x^2 (2), it takes the form

$$\frac{\partial q}{\partial t} - ((x^1)^2 + (x^2)^2 + 1)^2 \sum_{l,m=1}^{2} \frac{\partial}{\partial x^l} \frac{a(dx^l, dx^m)}{((x^1)^2 + (x^2)^2 + 1)^2} \frac{\partial u}{\partial x^m} = (A(x) - u(x))q - B(x)q^2;$$

cf. (1). The unknown function $q = q(t, x)$ corresponds to the density of the renewable resource under consideration at a point x of the sphere $\mathbb{S}^2$ at a time instant t, the metric a characterizes the resource diffusion, the function u is the control of its stationary (permanent) harvesting, and the coefficients A and B are the resource renewal and saturation rates of the environment.

Weak solutions, subsolutions, and supersolutions are defined in a conventional way [8, Ch. VI, §1, 5] and [9, §1.5]. In particular, a *weak solution* of equation (6) on the half-open interval $[T_0, T_1)$ is a function $q \in W([T_0, T_1); \mathbb{S}^2)$ such that $q^2 \in L^2([T_0, T_1) \times \mathbb{S}^2)$ and

$$\langle q, p\rangle(t) + \int_{T_0}^{t} \Big(\langle dq, dp\rangle_{L^2(T^*\mathbb{S}^2)} - \langle q, p'\rangle\Big)(\tau)d\tau = \langle q, p\rangle(0) + \int_{T_0}^{t} \langle (A-u)q - Bq^2, p\rangle(\tau)d\tau$$

for each $p \in C^\infty([T_0, T_1); \mathbb{S}^2)$ and $t \in [T_0, T_1)$. A weak solution q of equation (1.5) that takes a given initial value of the resource density,

$$q(T_0) = q_0, \quad q_0 \in L^\infty(\mathbb{S}^2), \quad q_0 \geqslant 0 \text{ a.e.}, \tag{7}$$

is called a *weak solution of the Cauchy problem* (6)*,* (7) *on* $[T_0, T_1)$.

In the presentation below, all solutions, subsolutions, and supersolutions are assumed to be weak, and the adjective "weak" is omitted for brevity.

3. PERIODIC IMPULSE HARVESTING AND CONTROLLED SOLUTIONS

3.1. Periodic Impulse Harvesting

The periodic impulse harvesting of a renewable resource is mathematically modeled by the solution q of the Cauchy problem (6), (7) with the additionally imposed conditions

$$q(kT) = sq(kT-), \;\; k = 1, 2, \ldots. \tag{8}$$

Here $T \in (0, +\infty)$ is a given period, and the measurable factor s, $0 \leqslant s \leqslant 1$ *a.e.*, characterizes the impulse harvesting rate. The solution of problem (6), (8) is a function $q \in L^\infty([0, +\infty) \times \mathbb{S}^2)$ that resolves equation (6) on $[kT, (k+1)T)$, has the left-hand limit values $q(kT-)$, and satisfies *a.e.* equalities (8). If for $T_0 = 0$ this solution takes *a.e.* the initial value (7), then it represents the solution of problem (6), (7), (8). The solution of problem (6), (8) is said to be *periodic* if

$$q(t + T) = q(t), \;\; t \in [0, +\infty). \tag{9}$$

We define the *admissible sets* $\mathfrak{U}$ *and* $\mathfrak{S}$ *of stationary and impulse controls*

$$\begin{gathered} \mathfrak{U} = \{u \in L^\infty(\mathbb{S}^2) \mid U_1 \leqslant u \leqslant U_2\}, \\ \mathfrak{S} = \{e^{-\beta v} \mid v \in L^\infty(\mathbb{S}^2), V_1 \leqslant v \leqslant V_2, \langle 1, v\rangle \leqslant E\}, \end{gathered} \tag{10}$$

where $U_1, U_2, V_1, V_2, \beta \in L^\infty(\mathbb{S}^2)$ and $E \in [0, +\infty)$. Here U_1 and U_2 characterize the constraints on the possible density of stationary resource harvesting, E is the admissible harvesting effort, and the limits V_1 and V_2 describe the minimum technically feasible density of impulse harvesting and its maximum possible density given the available physical capacity of the environment and ecological constraints. In essence, $V_1(x)$ and $V_2(x)$ are the minimum and maximum efforts that can be applied at a point x to achieve the goals. The impulse factor form $s = e^{-\beta(x)v(x)}$ in (8) stems from the search theory [21–23]. The factor $\beta(x)$ in the exponent characterizes the complexity of detecting and extracting the resource at a point $x \in \mathbb{S}^2$, and $v(x)$ is the effort applied.

Remark 1. As is easily checked, the sets of admissible stationary $\mathfrak{U}$ and impulse $\mathfrak{S}$ controls (10) are convex, closed in $L^2(\mathbb{S}^2)$, and bounded in $L^\infty(\mathbb{S}^2)$. Since the space $L^2(\mathbb{S}^2)$ is reflexive, by the Eberlein–Šmulyan theorem, the sets bounded in the norm $\|\cdot\|_{L^2(\mathbb{S}^2)}$ are sequentially weakly precompact [24, App. to Ch. V, §4]. In addition, each convex and closed subset of $L^2(\mathbb{S}^2)$ is weakly closed [25, Sec. 2.9]. Therefore, the sets of admissible controls $\mathfrak{U}$ and $\mathfrak{S}$ are sequentially weakly compact. A subset in $L^2(\mathbb{S}^2)$ is weakly compact if and only if it is sequentially weakly compact, and sequentially weakly precompact sets are norm-bounded [25, Sec. 2.9]. Therefore, the sets $\mathfrak{U}$ and $\mathfrak{S}$ are weakly sequentially compact in $L^2(\mathbb{S}^2)$ [24, App. to Ch. V, §4].

Remark 2. Obviously, $q = 0$ is a periodic subsolution of problem (6), (7), (8). Due to the constraints imposed above on equation (6) and the admissible controls, $B \geqslant B_0 > 0$ and $0 \leqslant s \leqslant 1$ *a.e.* Hence, the constant function $q = c$ is a periodic supersolution of problem (6), (7), (8) for $c \geqslant Q(\| q_0 \|_{L^\infty(\mathbb{S}^2)})$, where

$$Q : \mathbb{R} \ni r \mapsto \max\left\{r, \frac{1}{B_0}(\|A\|_{L^\infty(\mathbb{S}^2)} + \max\{\|U_1\|_{L^\infty(\mathbb{S}^2)}, \|U_2\|_{L^\infty(\mathbb{S}^2)}\})\right\} \in \mathbb{R}. \tag{11}$$

3.2. Controlled Solutions

The solutions $q = q(t; q_0, u, s)$ of problem (6), (7), (8) and the periodic solutions $q = q(t; u, s)$ of problem (6), (8) with admissible controls $u \in \mathfrak{U}$ and $s \in \mathfrak{S}$ will be called *controlled solutions*. They possess the following properties.

Theorem 1. *Assume that the metric a is measurable and satisfies the estimate* (4) *and the coefficients $A, B \in L^\infty(\mathbb{S}^2)$ and $B \geqslant B_0$ a.e. for some $B_0 \in (0, +\infty)$. Then:*

(a) *For any $u \in \mathfrak{U}$, $s \in \mathfrak{S}$, there exists a unique controlled solution $q = q(t; q_0, u, s)$. In addition, $q \in C([(k-1)T, kT); L^2(\mathbb{S}^2))$, $k = 0, 1, \ldots,$ and*

$$0 \leqslant q(t; q_0, u, s) \leqslant Q\left(\| q_0\|_{L^\infty(\mathbb{S}^2)}\right), \quad t \in [0, +\infty), \tag{12}$$

where Q is the function (11), *and for any $\varepsilon \in (0, T)$ there exists a number α, $0 < \alpha \leqslant 1$, such that*

$$q \in C^\alpha\left(\bigcup_{k=1}^{\infty}[(k-1)T + \varepsilon, kT) \times \mathbb{S}^2\right).$$

(b) *If sequences $\{q_m\} \subseteq L^\infty(\mathbb{S}^2)$, $\{u_m\} \subseteq \mathfrak{U}$, and $\{s_m\} \subseteq \mathfrak{S}$ weakly converge in $L^2(\mathbb{S}^2)$, i.e., $q_m \rightharpoonup q_0$, $u_m \rightharpoonup u_0$, and $s_m \rightharpoonup s_0$, and $q_m \geqslant 0$ and $q_m \neq 0$ a.e., then the weak convergence*

$$\lim_{m\to+\infty} q\,(\cdot\,; q_m, u_m, s_m) = q\,(\cdot\,; q_0, u_0, s_0)$$

holds in the spaces $L^2([0, NT); W^{1,2}(\mathbb{S}^2))$ for any $N = 1, 2, \ldots$ and in the norms $\|\cdot\|_{C(\cup_{k=1}^N[(k-1)T+\varepsilon, kT)\times\mathbb{S}^2)}$ for any $\varepsilon \in (0, T)$.

(c) *For any $u \in \mathfrak{U}$ and $s \in \mathfrak{S}$, there exists a unique controlled periodic solution $q = q_\infty(t; u, s)$ such that*

$$\lim_{t\to+\infty} \|q(t; q_0, u, s) - q_\infty(t; u, s)\|_{L^\infty(\mathbb{S}^2)} = 0, \quad \|q_0\|_{L^\infty(\mathbb{S}^2)} > 0.$$

(d) *If sequences* $\{u_m\} \subseteq \mathfrak{U}$ *and* $\{s_m\} \subseteq \mathfrak{S}$ *weakly converge in* $L^2(\mathbb{S}^2)$, *i.e.,* $u_m \rightharpoonup u_0$ *and* $s_m \rightharpoonup s_0$, *then the periodic solutions from item* (c) *have the weak convergence*

$$\lim_{m\to+\infty} q_\infty(\cdot\,; u_m, s_m) = q_\infty(\cdot\,; u_0, s_0)$$

in the space $L^2((0,T); W^{1,2}(\mathbb{S}^2))$ *and in the norms* $\|\cdot\|_{C([\varepsilon,T)\times\mathbb{S}^2)}$ *for any* $\varepsilon \in (0,T)$.

The proof is given in subsection 4.5.

Remark 3. There exist at most two periodic solutions q of problem (6), (8). According to Remark 2, one of them is the trivial solution $q = 0$. If $q_\infty = 0$, then by Theorem 1 the other disappears; if $q_\infty \neq 0$, then the third does the same.

4. PROBLEM STATEMENT, THE MAIN RESULT, AND FINDINGS

4.1. Problem Statement

According to assertion (a) of Theorem 1, the following functional is well-defined for the admissible sets of stationary $\mathfrak{U}$ and impulse $\mathfrak{S}$ controls (10):

$$F : \{q_0 \in L^\infty(\mathbb{S}^2) | q_0 \geqslant 0 \text{ a.e.}\} \times \mathfrak{U} \times \mathfrak{S} \ni (q_0, u, s)$$
$$\longmapsto \overline{\lim_{t\to+\infty}} \frac{1}{t}\left(\int_0^t \langle q(\tau; q_0, u, s), u\rangle d\tau + \sum_{0<kT\leqslant t} \langle q(kT-; q_0, u, s), 1 - s\rangle\right) \in \mathbb{R}, \tag{13}$$

where $q = q(t; q_0, u, s)$ is a controlled solution. Its value is the time-averaged sum of the stationary (first term) and impulse (second term) resource harvestings.

Let us pose the following problem: *It is required to establish the existence of stationary* $u_0 \in \mathfrak{U}$ *and impulse* $s_0 \in \mathfrak{S}$ *controls that maximize the functional* F (13), *and investigate the impact of the initial value* q_0 (7) *on* $F(q_0, u_0, s_0)$, *cf.* [26] *and* [27].

4.2. The Main Result

Using Theorem 1, we provide a comprehensive solution of this problem. Namely, the following result is true; cf. [28].

Theorem 2. *Assume that all conditions of Theorem 1 are satisfied. Then:*

(a) *For any initial values* q_0 (7), $\|q_0\|_{L^\infty(\mathbb{S}^2)} > 0$, *and admissible controls* $u \in \mathfrak{U}$, $s \in \mathfrak{S}$, *we have the equality*

$$F(q_0, u, s) = F(q_\infty(0; u, s), u, s) = \frac{1}{T}\left(\int_0^T \langle q_\infty(\tau; u, s), u\rangle d\tau + \langle q_\infty(T-; u, s), 1 - s\rangle\right). \tag{14}$$

(b) *If sequences* $\{u_m\} \subseteq \mathfrak{U}$ *and* $\{s_m\} \subseteq \mathfrak{S}$ *weakly converge in* $L^2(\mathbb{S}^2)$, *i.e.,* $u_m \rightharpoonup u_0$ *and* $s_m \rightharpoonup s_0$, *and a sequence* $\{q_m\} \subseteq L^\infty(\mathbb{S}^2)$ *is such that* $q_m \geqslant 0$ *and* $q_m \neq 0$ *a.e., then*

$$\lim_{m\to+\infty} F(q_m, u_m, s_m) = F(q_\infty(0; u_0, s_0), u_0, s_0).$$

(c) *The functional* F (13) *is bounded and its supremum is achieved at admissible controls* $u_0 \in \mathfrak{U}$ *and* $s_0 \in \mathfrak{S}$ *so that*

$$\sup F(q_0, u, s) = F(q_\infty(0;\, u_0, s_0), u_0, s_0).$$

Proof.

(a) Clearly, the value of the functional $F(q_0, u, s)$ will not change when replacing the zero lower limits of integration and summation in its definition (13) by any $T_0 \in [0, +\infty)$. Next, for controlled solutions $q = q(t; q_0, u, s)$ of problem (6), (7), (8) and a periodic solution $q = q_\infty(t; u, s)$ of problem (6), (8), we have

$$\left| \int_{T_0}^{t} \langle q(\tau) - q_\infty(\tau), u \rangle d\tau + \sum_{T_0 < kT \leqslant t} \langle q(kT-) - q_\infty(kT-), 1 - s \rangle \right|$$
$$\leqslant \left(t \parallel u \parallel_{L^\infty(\mathbb{S}^2)} \sup_{\tau \geqslant T_0} \|q(\tau) - q_\infty(\tau)\|_{L^\infty(\mathbb{S}^2)} + [t] \|1 - s\|_{L^\infty(\mathbb{S}^2)} \sup_{kT \geqslant T_0} \|q(kT-) - q_\infty(kT-)\|_{L^\infty(\mathbb{S}^2)} \right)$$
$$\leqslant t \Big(\max\{\|U_1\|_{L^\infty(\mathbb{S}^2)}, \|U_2\|_{L^\infty(\mathbb{S}^2)}\} + 1 \Big) \sup_{\tau \geqslant T_0} \|q(\tau) - q_\infty(\tau)\|_{L^\infty(\mathbb{S}^2)}, \qquad t \in [T_0, +\infty).$$

By assertion (c) of Theorem 1, it follows that $|F(q_0, u, s) - F(q_\infty(0; u, s), u, s)| = 0$. Due to definition (13), $F(q_\infty(0; u, s), u, s)$ equals the right-hand side of equality (14).

(b) According to assertion (a), we have

$$F(q_0, u_m, s_m) = \frac{1}{T} \left(\int_0^T \langle q_\infty(\tau; u_m, s_m), u_m \rangle d\tau + \langle q_\infty(T-; u_m, s_m), 1 - s_m \rangle \right).$$

By assertion (d) of Theorem 1, it is possible to pass to the limit on the right-hand side of this expression as $m \to +\infty$ [29, Ch. 1, §5]. As a result, in view of (14), we arrive at the desired conclusion.

(c) There exist sequences of initial values $\{q_m\} \subseteq L^\infty(\mathbb{S}^2)$ and admissible controls $\{u_m\} \subseteq \mathfrak{U}$ and $\{s_m\} \subseteq \mathfrak{S}$ such that

$$\sup F(q_0, u, s) = \lim_{m \to +\infty} F(q_m, u_m, s_m).$$

Due to Remark 1, the sets of admissible controls $\mathfrak{U}$ and $\mathfrak{S}$ are sequentially weakly compact in $L^2(\mathbb{S}^2)$. Hence, there exist subsequences $\{u_{m_l}\}$ and $\{s_{m_l}\}$ that weakly converge in $L^2(\mathbb{S}^2)$, i.e., $u_{m_l} \rightharpoonup u_0 \in \mathfrak{U}$ and $s_{m_l} \rightharpoonup s_0 \in \mathfrak{S}$. By assertion (b), we obtain

$$\sup F(q_0, u, s) = \lim_{m \to +\infty} F(q_m, u_m, s_m) = F(q_\infty(0; u_0, s_0), u_0, s_0).$$

The proof of Theorem 2 is complete.

4.3. Findings

According to assertion (c) of Theorem 1, after choosing admissible stationary and impulse controls, the renewable resource density will uniformly tend to a unique limit state for any nonzero initial values. According to assertion (c) of Theorem 2, admissible controls can be chosen so that for each exploitation cycle, the amount of resource harvesting coincides with the maximum possible time-averaged amount of resource harvesting. In other words, with the optimal control of renewable resource exploitation, any nonzero initial resource density will tend to a limiting state ensuring the maximum of resource harvesting in one exploitation cycle.

5. PROOF OF THEOREM 1

5.1. Auxiliary Assertions

According to Remark 1, the subsolution of problem (6), (7), (8) is the zero function $q = 0$, and the supersolution is the constant function $q = Q(\|q_0\|_{L^\infty(\mathbb{S}^2)})$. Therefore, the known results for second-order semilinear parabolic equations on a sphere [30–32] imply the following.

Lemma 1. *Assume that all conditions of Theorem 1 are satisfied. Then for each $u \in L^\infty(\mathbb{S}^2)$ there exists a unique solution $q = q(\cdot\,; q_0, u)$ of problem (6), (7) on the half-open interval $[T_0, +\infty)$. Moreover, $q \in C([T_0, +\infty); L^2(\mathbb{S}^2))$, $0 \leqslant q(t) \leqslant Q(\|q_0\|_{L^\infty(\mathbb{S}^2)})$ a.e. for $t \in [T_0, +\infty)$, and for each $\varepsilon > 0$ it is possible to find $\alpha = \alpha(\varepsilon, \|q\|_{L^\infty([T_0,+\infty))\times(\mathbb{S}^2)})$, $0 < \alpha \leqslant 1$, and $C = C(\varepsilon, \|q\|_{L^\infty([T_0,+\infty))\times\mathbb{S}^2}) \geqslant 0$ such that $q \in C^\alpha([T_0+\varepsilon, +\infty)) \times \mathbb{S}^2$ and $\|q\|_{C^\alpha([T_0+\varepsilon,+\infty))\times(\mathbb{S}^2)} \leqslant C$.*

In addition, we have the following fact.

Lemma 2. *Assume that all conditions of Theorem 1 are satisfied. If sequences $\{q_m\} \subseteq L^\infty(\mathbb{S}^2)$ and $\{u_m\} \subseteq \mathfrak{U}$ weakly converge in $L^2(X)$, i.e., $q_m \rightharpoonup q_0$ and $u_m \rightharpoonup u_0$, then the solutions $q = q(t; q_m, u_m)$ of the Cauchy problem (6), (7) have the weak convergence*

$$\lim_{m\to+\infty} q(\cdot\,;\ q_m, u_m) = q(\cdot\,;\ q_0, u_0)$$

in $L^2([T_0, T_1); W^{1,2}(X))$ and in the norms $\|\cdot\|_{C([T_0+\varepsilon,T_1)\times X)}$ for any $\varepsilon \in (0, T_1 - T_0)$.

Proof. By assertion (a) of Theorem 1, for $m = 1, 2, \ldots$ there exists a unique controlled solution $q(t; q_m, u_m)$ on the half-open interval $[T_0, T_1)$. Since the sequence $\{\|q_m\|_{L^\infty(\mathbb{S}^2)}\}$ is bounded (see Remark 1), we obtain

$$0 \leqslant q(t; q_m, u_m) \leqslant Q\left(\sup_{(m=0,1,\ldots} \|q_m\|_{L^\infty(\mathbb{S}^2)}\right), \quad t \in [T_0, T_1), \quad m = 1, 2, \ldots.$$

Therefore, based on the a priori estimates for the solutions of linear second-order parabolic equations [8, ch. VI, §1] and [9, §1.5], there exists a constant C_1 such that

$$\|q(\cdot\,;\ q_m, u_m)\|_{L^2\left((T_0,T_1);W^{1,2}(X)\right)} \leqslant C_1, \quad m = 1, 2, \ldots; \tag{15}$$

by assertion (a) of Theorem 1, for $\varepsilon \in (0, T_1 - T_0)$ it is possible to find C_2 and $0 < \alpha \leqslant 1$ such that

$$\|q(\cdot\,;\ q_m, u_m)_{|[T_0+\varepsilon,T_1)\times X}\ \|_{C^\alpha([T_0+\varepsilon,T_1)\times X)} \leqslant C_2, \quad m = 1, 2, \ldots. \tag{16}$$

Due to (15) and the Eberlein–Šmulyan theorem (see [24, App. to Ch. V, §4], the sequence $\{q(\cdot\,; q_m, u_m)\}$ is sequentially weakly precompact in $L^2((T_0, T_1); W^{1,2}(X))$ because this space is reflexive [17, Ch. III, §1]. In turn, due to (16) and the Arzelá–Ascoli theorem, the sequence $\{q(\cdot\,; q_m, u_m)_{|[T_0+\varepsilon,T_1)\times X}\}$ is sequentially precompact in the norms $\|\cdot\|_{C([T_0+\varepsilon,T_1)\times X)}$. Hence, from $\{q(\cdot\,; q_m, u_m)\}$ it is possible to select a subsequence $\{q(\cdot\,; q_{m_l}, u_{m_l})\}$ that weakly converges to the limit function

$$\tilde{q}(t) = \lim_{m\to+\infty} q(t; q_{m_l}, u_{m_l}) \in L^\infty([T_0, T_1); L^\infty(X))$$

in $L^2((T_0, T_1); W^{1,2}(X))$ and in the norms $\|\cdot\|_{C([T_0+\varepsilon,T_1)\times X)}$ for any $\varepsilon \in (0, T_1 - T_0)$.

For $q_0 = q_{m_l}$ and $u = u_{m_l}$, the solution of problem (6), (7) is defined by

$$\langle q(\cdot\,; q_{m_l}, u_{m_l}), p\rangle(t) + \int_{T_0}^{t} \Big(\langle dq(\cdot\,; q_{m_l}, u_{m_l}), dp\rangle_{L^2(T^*\mathbb{S}^2)} - \langle q(\cdot\,; q_{m_l}, u_{m_l}), p'\rangle\Big)(\tau)d\tau$$
$$= \langle q_{m_l}, p\rangle(0) + \int_{T_0}^{t} \langle (A - u_{m_l})q(\cdot\,; q_{m_l}, u_{m_l}) - Bq^2(\cdot\,; q_{m_l}, u_{m_l}), p\rangle(\tau)d\tau.$$

Passing to the limit as $l \to +\infty$ [29, Ch. 1, §5] yields

$$\langle \tilde{q}, p\rangle(t) + \int\limits_{T_0}^{t} \left(\langle d\tilde{q}, dp\rangle_{L^2(T^*\mathbb{S}^2)} - \langle \tilde{q}, p'\rangle\right)(\tau)d\tau = \langle q_0, p(0)\rangle + \int\limits_{T_0}^{t} \langle (A-u_0)\tilde{q} - B\tilde{q}^2, p\rangle(\tau)d\tau,$$

i.e., the limit function $\tilde{q}$ is the solution of problem (6), (7) on $[T_0, T_1)$ with the initial value q_0 and the stationary control u_0. By assertion (a) of Theorem 1, the solution of problem (6), (7) is unique and, consequently, $\tilde{q}(t) = q(t; q_0, u_0)$. The proof of Lemma 2 is complete.

5.2. *Proof of Assertions* (a) *and* (b)

Assertion (a) is a corollary of Lemma 1.

Assertion (b) is established by induction on $N = 1, 2, \dots$. For $N = 1$, the desired result follows from Lemma 2 for $[T_0, T_1) = [0, T)$. Assume that it is true for $N \geqslant 1$. Then the sequence $\{q(NT-; q_m, u_m, s_m)\}$ converges to $q(NT-; q_0, u_0, s_0)$ in $\|\cdot\|_{C(X)}$; therefore, $\{s_m q(NT-; q_m, u_m, s_m)\}$ weakly converges to $s_0 q(NT-; q_0, u_0, s_0)$ in $L^2(X)$ [29, Ch. 1, §5]. By Lemma 2, for $[T_0, T_1) = [NT, (N+1)T)$, we arrive at the weak convergence

$$\lim_{m\to+\infty} q(\cdot\,;\, s_m q(NT-; q_m, u_m, s_m), u_m, s_m) = q(\cdot\,; s_0 q(NT-; q_0, u_0, s_0), u_0, s_0)$$

in $L^2((0,T); W^{1,2}(X))$ and in the norms $\|\cdot\|_{C([\varepsilon,T)\times X)}$ for any $\varepsilon \in (0, T)$. Thus, the desired result holds for $(N+1)$ as well, and the proof is complete.

5.3. *Proof of Assertion* (c)

We choose an arbitrary number $r \in (0, +\infty)$ and consider the closed *function interval*

$$[0, Q(r)]_{L^\infty(X)} = \{w \in L^\infty(X) | 0 \leqslant w \leqslant Q(r) \text{a.e.}\},$$

where Q is the function (11). By Lemma 1, the *Poincaré operator*

$$P^u_{[T_0,T_1)} : [0, Q(r)]_{L^\infty(X)} \ni w \mapsto q(T_1; w, u) \in C(X)$$

is well defined, where $q = q(t; w, u)$ is the solution of problem (6), (7) on the half-open interval $[T_0, +\infty)$ with the initial value $q_0 = w$ and the admissible control $u \in \mathfrak{U}$ (10); cf. [33, Ch. III, §21]. In addition,

$$0 = P^u_{[0,\frac{T}{2})}0, \quad P^u_{[0,\frac{T}{2})}Q(r) \leqslant Q(r), \quad 0 = P^u_{[\frac{T}{2},T)}0, \quad P^u_{[\frac{T}{2},T)}Q(r) \leqslant Q(r)$$

due to Remark 2 and the comparison principle for weak solutions [9, Sec. 2.1.2], and, consequently,

$$\begin{aligned} &P^u_{[0,\frac{T}{2})}([0, Q(r)]_{L^\infty(X)}) \subseteq [0, Q(r)]_{L^\infty(X)}, \\ &P^u_{[\frac{T}{2},T)}([0, Q(r)]_{L^\infty(X)}) \subseteq [0, Q(r)]_{L^\infty(X)}. \end{aligned} \tag{17}$$

For the admissible controls $s \in \mathfrak{S}$ (10), we have $0 \leqslant s \leqslant 1$ *a.e.*, therefore

$$s[0, Q(r)]_{L^\infty(X)} \subseteq [0, Q(r)]_{L^\infty(X)}. \tag{18}$$

Thus, the composition of the Poincaré operator and multiplication by s is well-defined:

$$S : [0, Q(r)]_{L^\infty(X)} \ni v \mapsto P^u_{[0,\frac{T}{2})} s P^u_{[\frac{T}{2},T)} v \in [0, Q(r)]_{L^\infty(X)}. \tag{19}$$

Obviously, 0 is an *equilibrium* for S, i.e., $S(0) = 0$, whereas $Q(r)$ a *super-equilibrium*, i.e., $S(Q(r)) \leqslant Q(r)$ [33, Ch. I, §1]. According to assertion (a) and the Arzelá–Ascoli theorem, the operator S is continuous and has a precompact image. By the comparison principle, S strongly preserves order on $[0, Q(r)]_{L^\infty(X)}$ [33, Ch. I, §1]. Due to the strict concavity of the right-hand side of equation (6), the operator S is strictly sublinear, i.e., $\beta S(v) < S(\beta v)$ for $v \in [0, Q(r)]_{L^\infty(X)} \backslash 0$ and $0 < \beta < 1$. Hence, for any $r \in (0, +\infty)$, S has a unique fixed point $v_0 = Sv_0$ on the closed interval $[0, Q(r)]_{L^\infty(X)}$ such that

$$\lim_{k\to\infty} \|S^k(v) - v_0\|_{L^\infty(X)} = 0 \tag{20}$$

for any $v \in [0, Q(r)]_{L^\infty(X)} \backslash 0$ [33, Ch. I, §5]. In view of the inclusions (17) and (18), the function

$$q_{\infty,0} = sP^u_{[\frac{T}{2},T)} v_0 \in [0, Q(r)]_{L^\infty(X)}; \tag{21}$$

since equation (6) is *autonomous* (all its coefficients do not depend on t), it follows that

$$\begin{aligned} sP^u_{[0,T)} q_{\infty,0} &= sP^u_{[0,T)} \left(sP^u_{[\frac{T}{2},T)} v_0 \right) = sP^u_{[\frac{T}{2},T)} \left(P^u_{[0,\frac{T}{2})} sP^u_{[\frac{T}{2},T)} v_0 \right) \\ &= sP^u_{[\frac{T}{2},T)} Sv_0 = sP^u_{[\frac{T}{2},T)} v_0 = q_{\infty,0}. \end{aligned}$$

Thus, $q_{\infty,0}$ is a fixed point of the operator $sP^u_{[0,T)}$. As $q_\infty(t; u, s)$ we choose the solution $q(t; q_{\infty,0}, u, s)$ of problem (6), (7), (8) with the initial value $q_{\infty,0}$ (21). By assertion (a), this solution exists, is unique, and satisfies the estimate $0 \leqslant q_\infty(t; u, s) \leqslant Q(r)$ on the half-open interval $[0, +\infty)$; moreover, it satisfies the periodicity condition (9) because $q_{\infty,0}$ is a fixed point with respect to the operator $sP^u_{[0,T)}$.

Let $q(t; q_0, u, s)$ be the solution of problem (6), (7), (8) with $q_0 \in [0, Q(r)]_{L^\infty(X)} \backslash 0$. Then

$$w(t) = \pm(q(t; q_0, u, s) - q_\infty(t; u, s))$$

satisfies the weak maximum principle on the half-open intervals $[kT, k(T + 1))$, $k = 1, 2, \ldots$ [8, Ch. VI, §7] and, consequently,

$$|q(t; q_0, u, s) - q_\infty(t; u, s)| \leqslant |q(kT; q_0, u, s) - q_\infty(kT; u, s)|, \quad t \in [kT, k(T + 1)).$$

Since $q(kT; q_0, u, s) = (sP^u_{[0,T)})^k q_0$ and $q_\infty(kT; u, s) = (sP^u_{[0,T)})^k q_{\infty,0}$, it follows that

$$\|q(t; q_0, u, s) - q_\infty(t; u, s)\|_{C(X)} \leqslant \left\| \left(sP^u_{[0,T)} \right)^k q_0 - \left(sP^u_{[0,T)} \right)^k q_{\infty,0} \right\|_{L^\infty(X)};$$

by the construction of S (19) and the fixedness of $q_{\infty,0}$ (21) with respect to $sP^u_{[0,T)}$, we obtain

$$\|q(t; q_0, u, s) - q_\infty(t; u, s)\|_{C(X)} \leqslant \left\| sP^u_{[\frac{T}{2},T)} S^{k-1} P^u_{[0,\frac{T}{2})} q_0 - sP^u_{[\frac{T}{2},T)} v_0 \right\|_{L^\infty(X)}, \tag{22}$$

$t \in [kT, k(T + 1))$, because $(sP^u_{[0,T)})^k = sP^u_{[\frac{T}{2},T)} \left(P^u_{[0,\frac{T}{2})} sP^u_{[\frac{T}{2},T)} \right)^{k-1} P^u_{[0,\frac{T}{2})}$. According to (17),

$$P^u_{[0,\frac{T}{2})} q_0 \in P^u_{[0,\frac{T}{2})}([0, Q(r)]_{L^\infty(X)} \backslash 0) \subseteq [0, Q(r)]_{L^\infty(X)} \backslash 0;$$

due to (22), (20), the continuous operator $sP^u_{[\frac{T}{2},T)}$, and the arbitrary choice of $r \in (0, +\infty)$, we finally arrive at assertion (a).

5.4. Proof of Assertion (d)

By Lemma 1, there exist constants C and α, $0 < \alpha \leqslant 1$, such that

$$\|q_\infty(T-; u_m, s_m)\|_{C^\alpha(X)} \leqslant C$$

uniformly in $m = 1, 2, \ldots$. According to the Arzelá–Ascoli theorem, it is therefore possible to select a subsequence $\{q_\infty(T-; u_{m_l}, s_{m_l})\}$ from $\{q_\infty(T-; u_m, s_m)\}$ that will converge in the norm $\|\cdot\|_{C(X)}$ to the limit function

$$q_T = \lim_{l\to+\infty} q_\infty(T-; u_{m_l}, s_{m_l}). \tag{23}$$

It suffices to establish the equality

$$q_T = q_\infty(T-; u_0, s_0) \tag{24}$$

regardless of the choice of $\{q_\infty(T-; u_{m_l}, s_{m_l})\}$. In this case, the entire sequence $\{q_\infty(T-; u_m, s_m)\}$ will converge to $q_\infty(T-; u_0, s_0)$ in the norm $\|\cdot\|_{C(X)}$ and $\{s_m q_\infty(T-; u_m, s_m)\}$ will weakly converge to $s_0 q_\infty(T-; u_0, s_0)$ in $L^2(X)$ [29, Ch. 1, §5]; in the final analysis, Lemma 2 will imply assertion (d) since q_∞ satisfies conditions (8) and (9).

By conditions (8) and (9), $q_\infty(0; u_{m_l}, s_{m_l}) = s_{m_l} q_\infty(T-; u_{m_l}, s_{m_l})$; hence,

$$q_\infty(t; u_{m_l}, s_{m_l}) = q(t; s_{m_l} q_\infty(T-; u_{m_l}, s_{m_l}), u_{m_l}, s_{m_l}), \quad t \in [0, T),$$

as the solution of problem (6), (7) is unique (Lemma 1). Due to (23), the subsequence $\{s_{m_l} q_\infty(T-; u_{m_l}, s_{m_l})\}$ weakly converges in $L^2(X)$ to $s_0 q_T$. According to Lemma 2, passing to the limit on the right-hand side of this equality yields

$$q_T = q(T-; s_0 q_T, u_0, s_0).$$

Thus, the solution $q = q(t; s_0 q_T, u_0, s_0)$ satisfies the periodicity condition

$$q(0; s_0 q_T, u_0, s_0) = s_0 q(T-; s_0 q_T, u_0, s_0),$$

i.e., is a periodic solution of problem (6), (8). Hence, considering Remark 3, either $q_\infty(T-; u_0, s_0) = q(T-; s_0 q_T, u_0, s_0)$ (making (24) valid) or $q_\infty(T-; u_0, s_0) > 0$, $q(T-; s_0 q_T, u_0, s_0) = 0$, which is equivalent to the conditions

$$\|q_\infty(T-; u_0, s_0)\|_{C(X)} > 0, \quad q_T = 0. \tag{25}$$

We proceed by contradiction, showing that under conditions (25), the assertion

$$\lim_{\substack{k\to+\infty \\ l\to+\infty}} q(kT-; q_0, u_{m_l}, s_{m_l}) = 0, \quad q_0 > 0, \tag{26}$$

and its negation are simultaneously false; see items 1) and 2) below.

1) Assume that conditions (25) hold and assertion (26) is true.

By assertion (a), for $\varepsilon > 0$ there exists a natural number $k_0 = k_0(\varepsilon)$ such that

$$\|q(kT-; q_0, u_0, s_0) - q_\infty(T-; u_0, s_0)\|_{C(X)} < \varepsilon, \quad k = k_0, k_0 + 1, \ldots.$$

By assertion (b), for $\varepsilon > 0$ and $k = 1, 2, \ldots$ there exists a natural number $l_0 = l_0(\varepsilon, k)$ such that

$$\|q(kT-; q_0, u_{m_l}, s_{m_l}) - q(kT-; q_0, u_0, s_0)\|_{C(X)} < \varepsilon, \quad l_0, l_0 + 1, \ldots.$$

Consequently, based on

$$\begin{aligned}\|q(kT-;q_0,u_{m_l},s_{m_l})\|_{C(X)} &\geqslant \|q_\infty(T-;u_0,s_0)\|_{C(X)}\\ &- \|q(kT-;q_0,u_0,s_0)-q_\infty(T-;u_0,s_0)\|_{C(X)}\\ &- \|q(kT-;q_0,u_{m_l},s_{m_l})-q(kT-;q_0,u_0,s_0)\|_{C(X)},\end{aligned}$$

for any $\varepsilon>0$, $k=k_0(\varepsilon),k_0(\varepsilon)+1,\ldots$ and $l=l_0(\varepsilon,k),l_0(\varepsilon,k)+1,\ldots$, we have

$$\|q(kT-;q_0,u_{m_l},s_{m_l})\|_{C(X)} \geqslant \|q_\infty(T-;u_0,s_0)\|_{C(X)}-2\varepsilon.$$

Choosing $\varepsilon=\frac{\|q_\infty(T-;u_0,s_0)\|_{C(X)}}{4}>0$ in accordance with (25), we derive the estimate

$$\|q(kT-;q_0,u_{m_l},s_{m_l})\|_{C(X)} \geqslant \frac{\|q_\infty(T-;u_0,s_0)\|_{C(X)}}{2},$$

which contradicts assertion (26). Thus, conditions (25) and assertion (26) lead to a contradiction.

2) Assume that conditions (25) hold and assertion (26) is false. Then for some initial $q_0>0$ (7), there exists a number $\delta_0>0$ such that, for $N=1,2,\ldots,$ it is possible to find numbers $k_0=k_0(N)\geqslant N$ and $l_0=l_0(N)\geqslant N$ for which

$$\left\|q(k_0(N)T-;q_0,u_{m_{l_0(N)}},s_{m_{l_0(N)}})\right\|_{C(X)} \geqslant \delta_0. \tag{27}$$

Due to (23) and (25), for any $\varepsilon>0$ there exists a number $l_1=l_1(\varepsilon)$ such that

$$\|q_\infty(T-;u_{m_l},s_{m_l})\|_{C(X)}<\varepsilon,\quad l=l_1,l_1+1,\ldots.$$

By assertion (a), for $\varepsilon>0$ and $l=1,2,\ldots$ there exists a natural number $k_1=k_1(\varepsilon,l)$ such that

$$\|q(kT-;q_0,u_{m_l},s_{m_l})-q_\infty(T-;u_{m_l},s_{m_l})\|_{C(X)}<\varepsilon,\quad k=k_1,k_1+1,\ldots.$$

Therefore, for $0<\delta\leqslant\delta_0$ and $l=l_1(\frac{\delta}{2}),l_1(\frac{\delta}{2})+1,\ldots$ and $k=k_1(\frac{\delta}{2},l),k_1(\frac{\delta}{2},l)+1,\ldots,$ we have

$$\begin{aligned}\left\|q\left(k_0(N)T-;q_0,u_{m_{l_0(N)}},s_{m_{l_0(N)}}\right)\right\|_{C(X)} &\|q(kT-;q_0,u_{m_l},s_{m_l})\|_{C(X)}\\ \leqslant \|q(kT-;q_0,u_{m_l},s_{m_l})&-q_\infty(T-;q_0,u_{m_l},s_{m_l})\|_{C(X)}\\ &+\|q_\infty(T-;q_0,u_{m_l},s_{m_l})\|_{C(X)}<\delta.\end{aligned} \tag{28}$$

From (27) and (28) it follows that, for $N=l_1(\frac{\delta}{2}),l_1(\frac{\delta}{2})+1,\ldots$ there exists a number

$$k_2=k_2(\delta,k_0(N),l_0(N))\in\left\{k_0(N),\ldots,k_1\left(\frac{\delta}{2},l_0(N)\right)-1\right\}$$

for which

$$\begin{aligned}&\left\|q\left(k_2(\delta,k_0(N),l_0(N))T-;q_0,u_{m_{l_0(N)}},s_{m_{l_0(N)}}\right)\right\|_{C(X)}\geqslant\delta,\\ &\left\|q\left((k_2(\delta,k_0(N),l_0(N))+k)T-;q_0,u_{m_{l_0(N)}},s_{m_{l_0(N)}}\right)\right\|_{C(X)}<\delta,\quad k=1,2,\ldots.\end{aligned} \tag{29}$$

Consider the sequence $\{q_N\}$ composed of

$$q_N=q(k_2(\delta,k_0(N),l_0(N))T-;q_0,u_{m_{l_0(N)}},s_{m_{l_0(N)}}),\quad N=l_1\left(\frac{\delta}{2}\right),l_1\left(\frac{\delta}{2}\right)+1,\ldots.$$

By assertion (a), there exist constants C and α, $0 < \alpha \leqslant 1$, such that $\|q_N\|_{C^\alpha(X)} \leqslant C$ uniformly in N.

Based on the Arzelá–Ascoli theorem, we select a subsequence $\{q_{N_\beta}\}$ from the sequence $\{q_N\}$ that converges in the norm $\|\cdot\|_{C(X)}$ to the limit function

$$q_{0,\infty} = \lim_{\beta\to+\infty} q_{N_\beta}.$$

In view of the first inequality in (29), $\|q_{0,\infty}\|_{C(X)} \geqslant \delta$. Since the sequence $\{s_{m_{l_0(N_\beta)}} q_{N_\beta}\}$ weakly converges in $L^2(X)$ to $s_0 q_{0,\infty}$ [29, Ch. 1, §5], by assertion (b), for an arbitrary number $\varepsilon > 0$ and $k = 1, 2, \ldots$ there exists $\beta_0 = \beta_0(\varepsilon, k)$ such that

$$\left\| q\Big(kT-; s_{m_{l_0(N_\beta)}} q_{N_\beta}, u_{m_{l_0(N_\beta)}} q_{N_\beta}, s_{m_{l_0(N_\beta)}}\Big) - q(kT-; s_0 q_{0,\infty}, u_0, s_0) \right\|_{C(X)} < \varepsilon \tag{30}$$

for $\beta = \beta_0, \beta_0 + 1, \ldots$. By assertion (c), for $\varepsilon > 0$ there exists $k_3 = k_3(\varepsilon)$ such that

$$\|q(kT-; s_0 q_{0,\infty}, u_0, s_0) - q_\infty(T-; u_0, s_0)\|_{C(X)} < \varepsilon, \quad k = k_3, k_3 + 1, \ldots. \tag{31}$$

Hence, the following results are the case. First, since

$$\begin{aligned}\left\| q\left(kT-; s_{m_{l_0(N_\beta)}} q_{N_\beta}, u_{m_{l_0(N_\beta)}}, s_{m_{l_0(N_\beta)}}\right)\right\|_{C(X)} &\geqslant \|q_\infty(T-; u_0, s_0)\|_{C(X)} \\ &- \|q(kT-; s_0 q_{0,\infty}, u_0, s_0) - q_\infty(T-; u_0, s_0)\|_{C(X)} \\ - \left\| q\Big(kT-; s_{m_{l_0(N_\beta)}} q_{N_\beta}, u_{m_{l_0(N_\beta)}}, s_{m_{l_0(N_\beta)}}\Big) \right.&\left. - q(kT-; s_0 q_{0,\infty}, u_0, s_0)\right\|_{C(X)},\end{aligned}$$

considering (31) and (30), for $k \geqslant k_3(\varepsilon)$ and $\beta \geqslant \beta_0(\varepsilon, k)$ we have

$$\left\| q\left(kT-; s_{m_{l_0(N_\beta)}} q_{N_\beta}, u_{m_{l_0(N_\beta)}}, s_{m_{l_0(N_\beta)}}\right)\right\|_{C(X)} \geqslant \|q_\infty(T-; u_0, s_0)\|_{C(X)} - 2\varepsilon. \tag{32}$$

Second, by the construction of q_N and the autonomous property of equation (6),

$$q\left(kT-; s_{m_{l_0(N_\beta)}} q_{N_\beta}, u_{m_{l_0(N_\beta)}}, s_{m_{l_0(N_\beta)}}\right) = q\left((k_2(\delta, k_0(N), l_0(N)) + k)T-; q_0, u_{m_{l_0(N)}}, s_{m_{l_0(N)}}\right),$$

and the second inequality in (29) gives

$$\left\| q\left(kT-; s_{m_{l_0(N_\beta)}} q_{N_\beta}, u_{m_{l_0(N_\beta)}}, s_{m_{l_0(N_\beta)}}\right)\right\|_{C(X)} < \delta, \quad k = 1, 2, \ldots. \tag{33}$$

From (32) and (33), for $k = k_3(\varepsilon)$ and $\beta = \beta_0(\varepsilon, k_3(\varepsilon))$, we derive the inequality

$$\|q_\infty(T-; u_0, s_0)\|_{C(X)} < \delta + 2\varepsilon.$$

With

$$\varepsilon = \frac{\|q_\infty(T-; u_0, s_0)\|_{C(X)}}{4} \quad \text{and} \quad \delta = \min\left\{\frac{\|q_\infty(T-; u_0, s_0)\|_{C(X)}}{4}, \delta_0\right\},$$

the first condition in (25) leads to the contradictory estimate $\|q_\infty(T-; u_0, s_0)\|_{C(X)} < 0$. Thus, conditions (25) and the negation of assertion (26) bring to a contradiction as well.

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